1.

$$K(k) = P(k|k-1)\, C^T \left[C P(k|k-1) C^T + R \right]^{-1}$$

$$R = 0 \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$$

$$\underline{K}(k) = \begin{pmatrix} P_{11}(k|k-1) & P_{12}(k|k-1) \\ P_{12}(k|k-1) & P_{22}(k|k-1) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} P_{11}(k|k-1) & P_{12}(k|k-1) \\ P_{12}(k|k-1) & P_{22}(k|k-1) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^{-1}$$

$1\times 2 \quad 2\times 2 \quad 2\times 1$

$$\underline{K}(k) = \begin{pmatrix} \dfrac{P_{11}(k|k-1) + P_{12}(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} \\[2ex] \dfrac{P_{12}(k|k-1) + P_{22}(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} \end{pmatrix}$$

2.

$$P(k|k-1) = F\, P(k-1|k-1)\, F^T + Q$$

$$P(k|k-1) = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} P_{11}(k-1|k-1) & P_{12}(k-1|k-1) \\ P_{12}(k-1|k-1) & P_{22}(k-1|k-1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P(k|k-1) = \begin{pmatrix} P_{11}(k-1|k-1) & J P_{12}(k-1|k-1) \\ J P_{12}(k-1|k-1) & J^2 P_{22}(k-1|k-1) + 1 \end{pmatrix}$$

3).
$$P(k|k) = \left(I - K C \right) P(k|k-1).$$

$$P(k|k) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \right] \begin{pmatrix} P_{11}(k|k-1) & P_{12}(k|k-1) \\ P_{12}(k|k-1) & P_{22}(k|k-1) \end{pmatrix}$$

$$P(k|k) = \begin{pmatrix} \dfrac{P_{22}(k|k-1)P_{11}(k|k-1) - P_{12}^2(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} & \dfrac{P_{12}^2(k|k-1) - P_{11}(k|k-1)P_{22}(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} \\[2ex] \dfrac{P_{12}^2(k|k-1) - P_{11}(k|k-1)P_{22}(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} & \dfrac{P_{11}(k|k-1)P_{22}(k|k-1) - P_{12}^2(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)} \end{pmatrix}$$

$$P(k|k) = P_{11}(k|k) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

avec $$P_{11}(k|k) = \frac{P_{22}(k|k-1)P_{11}(k|k-1) - P_{12}^2(k|k-1)}{P_{11}(k|k-1) + 2P_{12}(k|k-1) + P_{22}(k|k-1)}.$$

4).
$$P_{11}(k|k) = \frac{\left[J^2 P_{22}(k-1|k-1) + 1\right] P_{11}(k-1|k-1) - J^2 P_{12}^2(k-1|k-1)}{P_{11}(k-1|k-1) + 2J P_{12}(k-1|k-1) + J^2 P_{22}(k-1|k-1) + 1}$$

D'après la forme de $P(k|k)$, on sait que :

$$P_{22}(k-1|k-1) = P_{11}(k-1|k-1) \qquad P_{12}(k-1|k-1) = -P_{11}(k-1|k-1).$$

Donc : $$P_{11}(k|k) = \frac{P_{11}(k-1|k-1)}{(J-1)^2 P_{11}(k-1|k-1) + 1}.$$

Si la limite de $\underset{(k\to\infty)}{P_{11}(k|k)}$ existe, elle vérifie $P_{11} = \dfrac{P_{11}}{(J-1)^2 P_{11} + 1}$

c'est-à-dire $P_{11}(J-1)^2 = 0$. Si $J \neq 1$, $P_{11} = 0$ et le filtre converge.